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APPROXIMATE METHOD FOR SOLVING RELAXATION PROBLEMS IN TERMS OF MATERIAL'S DAMAGABILITY UNDER CREEP

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The technology of thermoforming under creep and superplasticity conditions is finding increasing application in machine building for producing articles of a preset shape. After a part is made there are residual stresses in it, which lead to its warping. To remove residual stresses, moulded articles are usually exposed to thermal fixation, i.e., the part is held in compressed state at a certain temperature. Thermal fixation is simply the process of residual stress relaxation, following by accumulation of total creep in the material. Therefore the necessity to develop engineering methods for calculating the time of thermal fixation and relaxation of residual stresses to a safe level, not resulting in warping, becomes evident.

Below we present an approximate method of calculation of stress-strain state of a body during relaxation. Here we used a system of equations which describes a material's creep, simultaneously taking into account accumulation of damages in it.

Let us consider an arbitrary body (a structure element) limited by a surface S and referred to Cartesian rectangular system of coordinates x_k (k = 1, 2, 3). We assume that a part of the surface is free from external loads T_i (i = 1, 2, 3). Then

$$\sigma_{ij}\nu_j = 0. \tag{1}$$

On the other part of the surface S_u ($S = S_T + S_u$) the constants of displacement

$$u_i(x_k, t) = u_i^*(x_k) \tag{2}$$

and, consequently, of velocity $\dot{u}_i = 0$ are given. We also assume that the mass forces G_i are equal to zero. Then

$$\partial\sigma_{ij}/\partial x_j = 0. \tag{3}$$

Here σ_{ij} are the components of the stress tensor; u_i are the components of the displacement vector; the point denotes the time derivative; and ν_j are direction cosines of the external normal to the body's surface at the point under consideration.

The creep problems with conditions (1)-(3) we will call according to [1, 2] the relaxation problems. Here we assume that the components of the total strain velocity tensor $\dot{\epsilon}_{ij}$ is the sum of the components of the elastic strain velocity tensors $\dot{\varepsilon}_{ij}$ and total creep $\dot{p}_{ij}$ and connected with the components of the displacement velocity vector by the Cauchy relations

$$2\dot{\vartheta}_{ij} = \partial\dot{u}_i/\partial x_j + \partial\dot{u}_j/\partial x_i, \tag{4}$$

here for ε_{ij} Hooke's law ($\varepsilon_{ij} = 3s_{ij}/2E$) is valid, while for the total creep velocity the following law is valid [2, 3]:

$$\dot{p}_{ij} = \frac{B_1 S_2^{(n+1)/2}}{(1-\omega)^m}\frac{s_{ij}}{2S_2}. \tag{5}$$

Here E is the modulus of elasticity; $s_{ij} = \sigma_{ij} - \sigma_{kk}\delta_{ij}/3$; δ_{ij} is Kronecker's symbol; $S_2 = s_{ij}s_{ij}/2$ is the second invariant of the stress tensor; B_1, n, m are the creep characteristics of the material; and ω is the parameter of the material's damagability for which the kinetic equation has the form [2, 3]

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$$\dot{\omega} = B_2 S_2^{(g+1)/2}/(1-\omega)^m, \ \omega(x_k,0) = 0. \tag{6}$$

From (6) follows

$$\mu(x_k,t) = [1 - \int_0^t (m+1) B_2 S_2^{(g+1)/2} d\tau\,]^{1/(m+1)},$$

where $\mu(x_k, t) = 1 - \omega(x_k, t)$. From the condition $\omega = 1$ at a certain point with coordinates x_k^* (or the whole region) we can determine the time t_* when the body begins to be destroyed:

$$\int_0^{t_*} (m+1) B_2 S_2^{(g+1)/2} d\tau = 1.$$

Thus, the solution of a relaxation problem taking into account the material's damagability during creep is reduced to determining functions σ_{ij}, $\mathfrak{s}_{ij}$, p_{ij}, u_i, ω, t_*, which at any moment of time up to the beginning of the body's destruction satisfy the system of equations (3)-(6) with boundary conditions (1) and (2). A direct solution of this system is difficult because it is nonlinear and nonstationary. That is why approximate methods, in particular the principle of minimum of an additional power of deformations or the energy theorem [1], are used in solving relaxation problems. Below we prefer the energy theorem. It states that for each solid body the power of external forces is equal to the power of internal forces, i.e.,

$$\int_V (\sigma_{ij}\dot{\varepsilon}_{ij} + \sigma_{ij}\dot{p}_{ij}) dV = \int_V G_i \dot{u}_i dV + \int_S T_i \dot{u}_i dS. \tag{7}$$

It is evident that for relaxation problems ($G_i = 0$, $\dot{u}_i = 0$ on S_u, $T_i = 0$ on S_T) the right-hand side of (7) is zero.

At the initial moment of time $t = 0$ we have elastic distribution of stresses σ_{ij}^e in the body. Due to creep the stresses decrease with time, approaching zero. Solving relaxation problems without taking into account material damagability during creep, we usually seek for an approximate solution in the form [1, 2]

$$\sigma_{ij}^0(x_k,t) = \sigma_{ij}^e(x_k)\rho(t), \tag{8}$$

where relaxation factor after standard procedure takes the form

$$\rho(t) = [1 + (n-1)B_1 E t \int_V (S_2^e)^{(n+1)/2} dV / 3 \int_V S_2^e dV\,]^{-1/(n-1)}. \tag{9}$$

The damage accumulation process in a material considerably complicates the situation. Indeed, accumulated damages influence the creep velocity (this is reflected in the creep law (5)) and constantly facilitate redistribution of stresses [2]. Running ahead we note that the results of solving specific relaxation problems (for instance [4]) indicate an insignificant value of accumulated damages in the material and a constant reduction of the intensity of this accumulation process. One would think that in terms of this remark the damagability parameter in characteristic equations can be neglected. Nevertheless it seems unsuitable. Effectively, the damagability parameter is an "indicator" of a sort, whose value at any moment characterizes "structural state" of a material in the context of phenomenology. The less is the value the more is, naturally, a residual useful life of the material. Obviously, in similar problems there also appears a problem of determining such external temperature-force actions that as few as possible damages would accumulate in the material during preliminary strain (forming, forming with subsequent thermal fixation).

We propose to look for an approximate solution of a relaxation problem in terms of the process of damage accumulation in the material in the same form as in solving of the basic problem [5], namely

$$\sigma_{ij}(x_k,t) = \sigma_{ij}^0 f(x_k,t) + C(x_k,t)\delta_{ij}. \tag{10}$$

Here C is the hydrostatic component, which is determined with the use of the known method [5, 6] from the system of differential equations in partial derivatives

$$\frac{\partial C}{\partial x_j}\delta_{ij} = -\frac{\partial f}{\partial x_j}\sigma_{ij}^0 \tag{11}$$

with boundary condition C = 0 on the part of the surface S_T. As in solving the basic problem [5], the function

$$f(x_k,t) = [\mu(x_k,t)]^{m/n}/X(t). \tag{12}$$

Substituting (10) into (7) and (6), and taking into account (8), (9), (12), and Hooke's law, we obtain after simple transformations a system of equations for finding unknown functions X(t) and $\mu(x_k, t)$:

$$\frac{3}{2E}\int_V S_2^c\frac{d}{dt}[\mu^{m/n}\rho X^{-1}]^2 dV + (\rho X^{-1})^{n+1}\int_V B_1(S_2^c)^{(n+1)/2}\mu^{m/n}dV = 0; \tag{13}$$

$$\mu^{m/n} = \left[1 - \frac{\nu}{t_*^c}\int_0^t (\rho X^{-1})^{g+1}d\tau\right]^{\beta}. \tag{14}$$

Here

$$\beta = \frac{m}{n + m(n - g - 1)};\ \nu = \frac{n + m(n - g - 1)}{n(m + 1)};$$
$$t_*^c = [(m + 1)B_2(S_2^c)^{(g+1)/2}]^{-1};S_2^c = S_2^c(x_k).$$

In the general case the system of equations (13) and (14) admits of only numeric solution. We indicate one of the possible and simplest ways for solving this problem. Let us substitute (14) into (13). We obtain a differential equation with respect to function X(t), whose solution we will seek in the form of the series

$$X(t) = 1 + \sum_{k=1}^{\infty} a_k t^k.$$

It is easy to show that the sum of this series can be approximated by the expression

$$X(t) \approx (1 - \frac{1}{t_*^c}\int_0^t \rho^{g+1}d\tau)^{\gamma}, \tag{15}$$

where $\gamma = \beta\nu$; $t_*^c(\bar{x}_k)$; the coordinates $\bar{x}_k$ of the "median" by the volume point are found from the relation

$$[S_2^c(\bar{x}_k)]^{(n-1)/2} = \int_V (S_2^c)^{(n+1)/2}dV/\int_V S_2^c dV. \tag{16}$$

Substituting (15) into (14) and integrating, we obtain an a approximate expression for $\mu(x_k, t)$.

Thus, knowing X(t), $\mu(x_k, t)$, we determine the function $f(x_k, t)$ from (12). Combining $f(x_k, t)$, (8), and (9) we find from (10) the stressed state with an accuracy to the hydrostatic component $C(x_k, t)$. The hydrostatic component is calculated, as was pointed out above, from the system of differential equations in partial derivatives (11) with a boundary condition on the part of the surface S_T. The compatibility of this system is considered in each specific problem. If a system is incompatible, then the hydrostatic component is determined by minimizing a discrepancy in compatibility equations in the mean square.

From the known field of stresses σ_{ij} we find from (6) the distribution of the accumulated damages in a body at any moment of time, and from (5) and Hooke's law we find $\dot{p}_{ij}$, $\dot{\varepsilon}_{ij}$ and, consequently, the field of total strain velocities $\dot{\vartheta}_{ij} = \dot{p}_{ij} + \dot{\varepsilon}_{ij}$. From the known strain velocities we determine from (4) the components of the displacement velocity vector $\dot{u}_i$. To integrate system (4) it is necessary to fulfil six conditions of strain velocity continuity (Saint Venant continuity conditions). Obviously, these conditions are fulfilled only approximately, as in similar problems without regard to the process of damage accumulation in a material [1]. The components of the displacement vector are determined in the following way:

$$u_i(x_k,t) = u_i(x_k,0) + \int_0^t \dot{u}_i(x_k,\tau)d\tau.$$

As an example let us consider the problem of trimming of a thin walled gas-turbine engine disc of constant thickness. After mechanical treatment the disc, as a rule, has initial deflection w_0, which increases with time due to relaxation of residual self-balancing stresses, i.e., warping of the disc takes place. The disc under consideration has the following geometrical dimensions: the internal radius r_1 = 100 mm, the external radius r_2 = 300 mm, the disc thickness h = 3 mm. According to the existing norms the permissible deflection of the disc median plane must not exceed 3 mm ([w] = 3 mm). Analysis of the measurements of the warped discs' initial deflection shows that the latter is, as a rule, a function of only the radius r of the disc and exceeds considerably the permissible value, i.e., $\max_r w_0 > [w]$. Approximation of the initial deflection as a function of the radius has the form

$$w_0 = w_{max} \cos \frac{\pi r}{2r_2},$$

here $w_{max} \geq 3$ mm. Further we assume $w_{max} = 5$ mm.

In actual practice warping is removed in the following way. A finished disc, held in time, is placed into a die and loaded with a punch, so that the initial deflection at t = 0 becomes equal to zero at any r in the interval $r_1 \leq r \leq r_2$, i.e., $w_0(r, 0) = 0$. Then the loaded disc is placed into a reheating furnace and is held there in compressed state during a certain time t_1. In this case always $w(r, t) = 0$ and initial stresses σ_r^e, σ_φ^e, which are the sum of residual self-balancing stresses σ_r', σ_φ' and stresses σ_r'', σ_φ'' appearing as a result of loading the disc by a punch at t = 0, relax. After thermal fixation time t_1 the disc is unloaded, a residual deflection, if any, is measured and compared with the permissible one. Obviously, for a successful trimming of a disc it is necessary to know optimal time and thermal fixation temperature, for which purpose a relaxation problem should be solved.

We present the results of solving such problem. The disc considered here is made of titanium alloy BT9. The experiments on elastoplastic strain and strain during creep enabled us to determine the corresponding characteristics of a material in the temperature range from 400 to 650°C. For example, the temperature dependence of the modulus of elasticity has the form

$$E = (-0.01\, T + 12.30) \cdot 10^3,$$

where E, kg/mm^2; T, °C.

The thermal fixation temperature is agreed with the manufacturer and is set to be 550°C. At this temperature the creep characteristics in terms of the system of equations (5) are:

$$B_1 = 5.87 \cdot 10^{-9} (\text{kgsec/mm}^2)^{-n} \cdot \text{h}^{-1},\ n = g = 4,\ m = 11,$$
$$B_2 = 0.255 \cdot 10^{-9} (\text{kgsec/mm}^2)^{-(g+1)} \cdot \text{h}^{-1}.$$

The elastic field of stresses σ_r^e, σ_φ^e at t = 0 was determined, as was noted above, in the form

$$\sigma_r^e = \sigma_r' + \sigma_r'',\ \sigma_\varphi^e = \sigma_\varphi' + \sigma_\varphi''.$$

The field of residual self-balancing stresses was assumed to be uniformly distributed along the disc thickness, moreover for radial stress σ_r' the following approximation was used:

$$\sigma_r' = 10 \cdot \sin\left[\frac{\pi(r - r_1)}{r_2 - r_1}\right],$$

and shearing stress σ_φ' was found from the equilibrium equation

$$\frac{d\sigma_r'}{dr} + \frac{\sigma_r' - \sigma_\varphi'}{r} = 0.$$

The values of residual stresses, calculated with the help of the approximation indicated above, are in good agreement with check measurements.

It was suggested that during loading of disc at the initial moment of time t = 0 the Kirchhoff–Laves hypothesis of plane cross-sections is met and the strains ε_r, ε_φ are determined by the expressions

$$\varepsilon_r = \frac{du}{dr} + \frac{1}{2}\left(\frac{dw}{dr}\right)^2 + \frac{d^2w}{dr^2} z,$$
$$\varepsilon_\varphi = \frac{u}{r} + \frac{1}{r}\frac{dw}{dr} z,$$

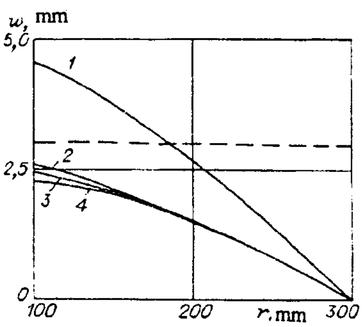

Fig. 1

where u(r) is the displacement along the r axis of the disc in the median plane and the z axis is perpendicular to the median plane. Loading of disc at t = 0 was simulated by the action of a uniformly distributed load q(r). The boundary conditions in this case can be written as $\sigma_r'' = 0$ for $r = r_1$, $u = 0$ for $r = r_2$. We think that Karman equations for axisymmetric plates [7] are valid:

$$D\frac{d}{dr}(\nabla^2 w) = \psi - \frac{h}{r}\frac{d\Phi}{dr}\frac{dw}{dr},$$

$$\frac{d}{dr}(\nabla^2\Phi) = \frac{E}{2r}\left(\frac{dw}{dr}\right)^2.$$

Here the cylindrical rigidity of the plate $D = Eh^3/12(1 - \nu^2)$ ($\nu = 1/2$); Φ is the Airy stress function; ψ is the load function; the operator $\nabla^2 = \frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right)$.

The ultimate system of Karman equations together with the Kirchhoff–Laves plane cross-section hypothesis and the indicated boundary conditions offer a way to determine the stress-strain state of a disc at the moment t = 0. Note that here and below in numerical calculation we reduced the boundary problem for an ordinary third-order differential equation to the boundary problem for a system of the ordinary first-order differential equations, which was solved by the Godunov method of orthogonal run [8]. Here we integrated by the Runge–Kutta method of the fourth order. We added up (algebraically) the stresses σ_r'', σ_φ'' thus determined with the residual stresses σ_r', σ_φ' and for the initial moment we obtained an elastic field of stresses σ_r^e, σ_φ^e.

For $t \geq 0$ we have $w(x_k, t) = 0$ and σ_r^e and σ_φ^e relax. We calculate stresses according to (10) at any moment of time. First, using σ_r^e and σ_φ^e we determine from (9) the relaxation factor $\rho(t)$ and from (8) we find the field of stresses σ_r^0 and σ_φ^0. From (15) and (16) we find X(t). Substituting X, σ_r^0, σ_φ^0 into (14) and integrating, we find $\mu(r, t)$. From (12) we find f(r, t) by the known X and μ. Knowing f(r, t), σ_r^0, σ_φ^0, we calculate from (10) $\sigma_r(r, t)$ and $\sigma_\varphi(r, t)$. The hydrostatic component in (10) is equal to zero because for the disc under consideration a plane stressed state is realized.

Let us dwell on the calculation of a residual deflection after the process of thermal fixation of disc and its elastic unspringing (unloading). For the moment $t = t_1$ from the known field of stresses $\sigma_r(r, t_1)$, $\sigma_\varphi(r, t_1)$ we find the distribution of the moments of deflection $M_r(r, t_1)$ and $M_\varphi(r, t_1)$ from the relations

$$M_r = \int_{-h/2}^{h/2}\sigma_r z dz, \quad M_\varphi = \int_{-h/2}^{h/2}\sigma_\varphi z dz.$$

From the equilibrium equation

$$\frac{dM_r}{dr} + \frac{M_r - M_\varphi}{r} = Q$$

we have the law of distribution of the shearing force $Q(r, t_1)$. At the time $t = t_1$ we carry out an elastic unloading of the disc, for which purpose the load $Q(r, t_1)$ should be "removed" in each point of the disc. The residual deflection we find from the differential equation

$$D \frac{d}{dr} (\nabla^2 w) = - Q(r,t_1)$$

with the boundary conditions that take into account a release of the disc contours $M_r = -M_r(r_1, t_1)$, $M_r = -M_r(r_2, t_1)$, $w(r_2, t_1) = 0$. As an illustration Fig. 1 shows deflections after unloading for different values of time of the disc blade fixation ($t_1 = 0$; 0.5; 2; lines 1-4 denote 5 h). The dashed line shows a maximum permissible disc deflection [2] = 3mm. It is clear from the figure that thermal fixation time $t_1 = 0.5$ h for the given initial deflections w_0 would already suffice for the residual deflections to be within permissible values after the disc release.

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